Translation-like Actions and Aperiodic Subshifts on Groups

Emmanuel Jeandel
LORIA, UMR 7503 - Campus Scientifique, BP 239
54 506 VANDOEUVRE-LÈS-NANCY, FRANCE
emmanuel.jeandel@loria.fr

27th August 2015

Abstract

It is well known that if G admits a f.g. subgroup H with a weakly aperiodic SFT (resp. an undecidable domino problem), then G itself has a weakly aperiodic SFT (resp. an undecidable domino problem). We prove that we can replace the property "H is a subgroup of G" by "H acts translation-like on G", provided H is finitely presented.

In particular:

- If G_1 and G_2 are f.g. infinite groups, then $G_1 \times G_2$ has a weakly aperiodic SFT (and actually a undecidable domino problem). In particular the Grigorchuk group has an undecidable domino problem.
- Every infinite f.g. p-group admits a weakly aperiodic SFT.

A subshift of finite type over a group G corresponds to a description of colorings of the vertices of its Cayley graph subject to local constraints. Even for harmless groups like $\mathbb{Z}^2$, it is possible to build [4] easily [12] subshifts with no periodic points. For this group, the domino problem, which consists in deciding if a subshift of finite type is empty, is even undecidable [4].

In this article, we are interested in which groups enjoy similar properties: In which groups can we build aperiodic subshifts of finite type, and which groups have an undecidable word problem. There are various definitions of "aperiodicity" on a group, and here we study weakly aperiodic subshifts: no coloring has a finite orbit.

Apart from the example of $\mathbb{Z}^2$, aperiodic subshifts have been built on Baumslag-Solitar Groups [1], on the free group [17] and on every group of nonlinear polynomial growth [2, 6]. In all these examples except the free group, this actually gives groups with an undecidable domino problem.

It is easy to see that if a f.g. group G has an aperiodic SFT, then every group that contains G also has an aperiodic SFT. In this statement, "contains" means subgroup containment. The goal of this article is to prove that this is

also true in a stronger sense. We say that H acts translation-like on G if, upto the deletion of some edges, some Cayley graph of G can be partitioned into copies of the Cayley graph of H.

We will then prove that if H is finitely presented, acts translation-like on G, and has an aperiodic SFT, then G also has an aperiodic SFT. As a corollary, we are able to show that the direct product of two infinite f.g. groups has an aperiodic SFT, and that any nonamenable f.g. group has an aperiodic SFT.

The main idea from this article comes from the work of Ballier and Stein [2], which explains (in particular) how to build aperiodic SFTs on $\mathbb{Z} \times G$ as soon as G is f.g infinite, by producing copies of $\mathbb{Z}$ inside G. We generalize this statement by searching for copies of other groups inside G. The main idea is that if G contains copies (in the sense of translation-like actions) of some group H, and if H is finitely presented, then we can find and describe these copies by a finite number of local constraints.

The idea is also reminiscent of Cohen [9] who proved that having a weakly aperiodic SFT is a quasi-isometry invariant for finitely presented groups. Two groups G and H are quasi-isometric if their Cayley graph look the same from a distance. This notion is somehow related to translation-like action, but there are differences: a quasi-isometry cannot distort the distances too much, and a translation-like action cannot identify two vertices of the Cayley Graph, so that these notions are ultimately different (The article [11] possibly provides an example of quasi-isometric groups G and H s.t. G does not act translation-like on H). The proof method used by Cohen is similar to ours: Cohen encode in a subshift in G local information that is sufficient to exhibit a quasiisometry from G to H. We do the same with translation-like actions, which is ultimately much simpler.

1 Definitions

We assume some familiarity with group theory and actions of groups. See [7] for a good reference on symbolic dynamics on groups.

All groups below are implicitely supposed to be finitely generated (f.g. for short).

The notion of a Cayley graph is used throughout this article to give some intuitions, but is not technically needed. If G is a group with generators S, the Cayley graph $C(G; S)$ is the graph with vertices G, and edges (g, gs) for $s \in S$.

The Baumslag-Solitar groups $B(m, n)$ is the group $B(m, n) = \langle a, b | ab^m a^{-1} = b^n \rangle$. When we speak of a Baumslag-Solitar group, we implicitely suppose that both m and n are nonzero (they might be negative). We will be interested mainly in the groups $B(1, n)$. $B(1, 1)$ is in particular the group $\mathbb{Z}^2$.

1.1 Symbolic dynamics on groups

Let A be a finite set and G a group. We denote by A^G the set of all functions from G to A. For $x \in A^G$, we write x_g instead of $x(g)$ for the value of x in g.

G acts on A^G by

$$(g \cdot x)_h = x_{g^{-1}h}$$

A *pattern* is a partial function P of G to A with finite support. The support of P will be denoted by $\mathrm{Supp}(P)$.

A *subshift* of A^G is a subset X of A^G which is topologically closed (for the product topology on A^G) and invariant under the action of G.

A subshift can also be defined in terms of forbidden patterns. If $\mathcal{P}$ is a collection of patterns, the subshift defined by P is

$$X_{\mathcal{P}} = \left\{ x \in A^G | \forall g \in G, \forall P \in \mathcal{P} \exists h \in \mathrm{Supp}(P), (g \cdot x)_h \neq P_h \right\}$$

Every such set is a subshift, and every subshift can be defined this way. If X can be defined by a finite set $\mathcal{P}$, X is said to be a subshift of finite type, or for short a SFT.

For a point $x \in X$, the stabilizer of x is $Stab(x) = \{g | g \cdot x = x\}$ A point x is strongly periodic if $Stab(x)$ is a subgroup of G of finite index. Equivalently, the orbit $G \cdot x$ of x is finite.

A subshift X is *weakly aperiodic* if it is nonempty and does not contain any strongly periodic point. In the remaining, we are interested in groups G which admit weakly aperiodic SFTs

A f.g. group G is said to have decidable domino problem if there is an algorithm that, given a description of a finite set of patterns $\mathcal{P}$, decides if $X_{\mathcal{P}}$ is empty. It is easy to see that f.g. groups with undecidable word problem have trivially an undecidable domino problem, so this question is mostly relevant for groups with a decidable word problem.

These two properties are related in the following way: If every (nonempty) SFT over G has a strongly periodic point (and G has decidable word problem), then G has decidable domino problem.

Finding which groups have a weakly aperiodic SFT, and which groups have decidable domino problems is the topic of this article.

We now summarize previous theorems:

Theorem 1.

- *$\mathbb{Z}$ does not admit a weakly aperiodic SFT and has decidable word problem*
- *Free groups of rank ≥ 2 admit weakly aperiodic SFTs [17] and have decidable word problem [14]*
- *The free abelian group $\mathbb{Z}^2$ [3], the Baumslag Solitar groups [1] admit weakly aperiodic SFTs, and have an undecidable word problem*
- *f.g. Nilpotent groups have weakly aperiodic SFTs and an undecidable word problem unless they are virtually cyclic [2, 6]*

We note in passing that these questions may be asked more generally for tilings of translation surfaces rather than coloring of groups, for example of $\mathbb{R}^2$, the hyperbolic plane [13], amenable spaces [5], or manifolds of intermediate growth [15]. How these results translate into aperiodic SFTs on groups is not clear. We will provide group versions of the two last theorems in this paper.

There are a few general statements on these questions which work as follow:

Theorem 2.

- *Let G, H be f. g. commensurable groups. Then G admits a weakly aperiodic SFT (resp. has an undecidable domino problem) if only if H does. [6]*
- *Let H be a f.g. normal subgroup of G f.g. If G/H admits a weakly aperiodic SFT (resp. has an undecidable domino problem), then G does.*
- *Let G, H be* finitely presented *groups that are quasi-isometric. Then G admits a weakly aperiodic SFT (resp. has an undecidable domino problem) if and only if H does. [9]*
- *Let G, H be two finitely presented groups that are quasi-isometric but not commensurable. Then G and H admit weakly aperiodic SFT [9]*

The relevant articles above [6, 9] only deal with weakly aperiodic SFTs (or strongly aperiodic SFTs) but the results about undecidable domino problems may be obtained in the same manner.

We conclude this review by the following easy result:

Proposition 1.1. *Let $H \subseteq G$ be f.g. groups. If H has a weakly aperiodic SFT (resp. an undecidable domino problem), then G does.*

Proof. Let X_H be a SFT on H.

$$X_H = \left\{x \in A^H | \forall h \in H, \forall P \in \mathcal{P} \exists p \in \mathrm{Supp}(P), (h \cdot x)_p \neq P_p \right\}$$

Taking the same forbidden patterns, we obtain a subshift X_G on G.

$$X_G = \left\{x \in A^G | \forall g \in G, \forall P \in \mathcal{P} \exists p \in \mathrm{Supp}(P), (g \cdot x)_p \neq P_p \right\}$$

If X_G is nonempty, then X_H is nonempty: The restriction of $x \in X_G$ to A^H is in X_H. If X_H is nonempty, then X_G is nonempty: Take $x \in X_G$, writee $G = HK$ for some transversal K and define $y_{hk} = x_h$.

This proves that G has an undecidable domino problem if H does.

Now if X_G has a point x with stabilizer N of finite index in G, then the restriction of x to A^H is a point of X_H with stabilizer that contains $H \cap N$, hence is of finite index in H. Hence G has a weakly aperiodic SFT if H does. $\square$

From all previous theorems, virtually cyclic groups are the only groups for which we are able to prove that they don't have aperiodic SFT, and virtually free groups the only groups for which we are able to prove that they have decidable domino problem. The conjectures below state these are the only cases:

Conjecture 1 ([2])**.** *A f.g. group G has a decidable domino problem iff it is virtually free.*

Conjecture 2 ([6])**.** *A f.g. group G has no weakly aperiodic SFT iff it is virtually cyclic.*

1.2 Translation-like actions

The concept of translation-like action was introduced by Whyte [20]. We will use here an alternative definition which is a compromise between the original definition from Whyte and Cor 5.2 in Seward [18].

Definition 1.2. *Let H and G be f.g groups. Let S_H be a generating set for H.*
We say that H acts translation-like on G if H right acts on G s.t:

- *The action is free: $g \circ h = g$ for some g implies $h = \lambda_H$.*
- *There exists a finite set S_G and a map $\phi : G \times S_H \to S_G$ s.t. $g \circ h = g\phi(g, h)$.*

Said otherwise, there exists a finite set S_G and a partial labelling of the edges of the Cayley graph $C(G, S_G)$ by elements of S_H s.t. the restrictions of $C(G, S_G)$ to the labelled edges is the disjoint union of copies of the Cayley graph of $C(H, S_H)$.

Note that the definition does not depend on the choice S_H of a generating set for H.

As hinted in [20, 18], translation-like actions generalize (for f.g. groups) subgroup containment: If H is a subgroup of G, then H acts translation-like on G. More generally:

Lemma 1.3. *Let G, H, N be f.g. groups.*
If H acts translation-like on N and N acts translation-like on G, then H acts translation-like on G.

Proof. This is obvious from the definition in terms of Cayley graphs.

For an actual proof, let $\square$ be the action of H on N (witnessed by the sets S_H and S_N) and $\circ$ be the action of N on G (witnessed by the sets S_N and S_G, recall we can take the same set S_N)

Write $G = K \circ N$ for some transversal K (which exists by freeness of the action) and define the action by $(k \circ i) \star h = k \circ (i \square h)$.

It is indeed an action. Furthermore, let $h \in S_H$ and $g \in G$. Write $g = k \circ i$. $(k \circ i) \star h = k \circ (i \square h) = k \circ (in)$ for some $n \in S_N$ as $\square$ is translation-like, and $k \circ (in) = (k \circ i)g'$ for some $g' \in S_G$ as $\circ$ is translation-like. Therefore $g \star h = gg'$ for some $g' \in S_G$. $\square$

2 The construction

Before going into the proof, we start with a warmup.

Let G be an f.g. infinite group and consider its Cayley graph $C(G;S)$ with respect to some finite set of generator S. Now obtain a subgraph of $C(G;S)$ by keeping only one outgoing edge and one incoming edge for each vertex. This subgraph $\mathcal{G}$ is therefore an union of biinfinite paths and circuits. Furthermore, it is always possible (if S has been chosen carefully) for $\mathcal{G}$ to consist only of biinfinite paths (This is nontrivial and comes from the fact that $\mathbb{Z}$ acts translation-like on any f.g. infinite group). In other words, $\mathcal{G}$ is the union of copies of $\mathbb{Z}$. The choice of an incoming and an outgoing edge can be simulated easily by a subshift of finite type, so that we have proven that for any f.g. infinite group G, there is a SFT S where every element of S somehow partitions the Cayley graph of G into copies of the Cayley graph of $\mathbb{Z}$ or of quotients of $\mathbb{Z}$, and some element of S partitions the Cayley graph of G into copies of the Cayley graph of $\mathbb{Z}$ only.

This construction is already sufficient to prove that $G_1 \times G_2$ admits an aperiodic SFT whenever G_1 and G_2 are f.g. infinite. To do better, we could try to embed something other than $\mathbb{Z}$, for example $\mathbb{Z}^2$, or the free group $\mathbb{F}_2$.

The general idea is as follows: Let H be a f.g. group. We want to build a SFT X over a group G s.t.

1. Each element of X somehow partitions the Cayley graph of G into copies of the Cayley graphs of H or of quotients of H.

2. There exist an element of X for which the partition correspond to copies of the Cayley graph of H itself.

The good notion to make the second point work is the concept of translation-like action. With a subshift of finite type, we can only test properties of the Cayley graph in a finite neighborhood of every point, this is the reason why we will require H to be finitely presented.

2.1 Some Definitions

We start with some notations. Let H be a finitely presented group that we see as a finitely presented *monoid* $H = \langle h_1 \dots h_n | R_1 \dots R_p \rangle$. Let $S = \{h_1 \dots h_n\}$. Let $\mathcal{R}$ be the set of relations seen as a subset of $S^\star \times S^\star$.

In the rare circumstances when it will be necessary to differentiate an element of $S^\star$ from the corresponding element of H, we will write h for an element of $S^\star$ and $\underline{h}$ for the corresponding element of H.

Let G be a f.g. group and S_G be a finite subset of G s.t. H acts translation like on G for this choice of S_G.

We denote by F the set of functions from S to S_G. Let Σ be a finite alphabet. In the remaining we are interested in points of $(\Sigma \times F)^G$. For $x \in (\Sigma \times F)^G$, we write $\sigma(x_g)$ for the Σ-component of x_g and $x_g(h_i)$ for the value of the function

in the F-component of x_g at h_i. In other words $\sigma(x_g)$ is a notation for $(x_g)_1$ and $x_g(h_i)$ a notation for $(x_g)_2(h_i)$.

A point x should be interpreted on the Cayley graph $C(G, S_G)$ of G. $\sigma(x_g)$ is the color at vertex g. $x_g(h_i)$ is the edge of $C(G, S_G)$ we have to follow from vertex g if we want to simulate going into direction h_i in the Cayley graph of H.

2.2 Preliminaries

For now on, we concentrate on the edges, and we will deal with the symbols later on. Let $x \in (\Sigma \times F)^G$. For g in G and $h \in S^\star$, we denote by $\Phi(g, x, h)$ the element of G obtained starting from g and following the edges (given by x) corresponding to h.

Formally Φ can be defined by :

- $\Phi(g, x, \epsilon) = g$ (where ϵ is the empty word of $S^\star$)
- $\Phi(g, x, h_i h) = \Phi(g x_g(h_i), x, h)$

Note that, for each element x, this defines an action of $S^\star$ into G, that is:

Fact 2.1. $\Phi(g, x, hh') = \Phi(\Phi(g, x, h), x, h')$.

Another way to see Φ is as follows (this can be proven by a straightword induction on the length of h):

Fact 2.2. $\Phi(g, g \cdot x, h) = g\Phi(\lambda_G, x, h)$.

The main object that will interest us in this section is the following subshift:

$$\begin{aligned} X &= \left\{x \in (\Sigma \times F)^G | \forall g \in G, \forall (h, h') \in \mathcal{R}, \Phi(\lambda_G, g \cdot x, h) = \Phi(\lambda_G, g \cdot x, h')\right\} \\ &= \left\{x \in (\Sigma \times F)^G | \forall g \in G, \forall (h, h') \in \mathcal{R}, \Phi(g, x, h) = \Phi(g, x, h')\right\} \end{aligned}$$

The two conditions are equivalent by the second fact.

Note that if we denote $Z = \left\{x \in (\Sigma \times F)^G | \forall (h, h') \in \mathcal{R}, \Phi(\lambda_G, x, h) = \Phi(\lambda_G, x, h')\right\}$, then $X = \{x | \forall g, g \cdot x \in Z\}$, which proves that X is indeed a subshift. As $\mathcal{R}$ and A are finite, whether $x \in Z$ depends only on the value of x on a finite neighborhood of λ_G, hence X is a SFT.

X is always well-defined, whether H acts translation-like on G or not. However X can be empty.

Now, by its definition, X has the following obvious property.

Fact 2.3. *Let $h, h' \in S^\star$ s.t. $\underline{h} = \underline{h'}$ (i.e. h and h' represent the same element in H).*

Then for $x \in X$, $\phi(g, x, h) = \phi(g, x, h')$.

In the remainder of this section, we introduce the key lemmas, which explain how any element of X can be seen as an element of Σ^H, and any element of Σ^H can be seen as an element of X with a specific property.

Definition 2.4. *Let $F : X \to \Sigma^H$ defined by $F(x)_{\underline{h}} = \sigma(x_{\Phi(\lambda,x,h)})$*

That is $F(x)_{\underline{h}}$ is the symbol we obtain starting from λ_G and following the direction given by h. As $x \in X$, this is well defined.

Lemma 2.5. *Let $x \in X$. Then for every $h \in H$, there exists $g \in G$ s.t. $h \cdot F(x) = F(g \cdot x)$.*

Proof. For $\underline{h} \in H$, take $g = \phi(\lambda, x, h)$. □

Lemma 2.6. *Let $y \in \Sigma^H$. Then there exists $x \in X$ s.t. $F(x) = y$ and for any $g \in G$ there exists $h \in H$ s.t. $h \cdot F(x) = F(g \cdot x)$*

Proof. By definition H acts translation like on G by $\circ$. Let $T \subseteq G$ be a set of representatives of this free action, that is for every $g \in G$ there are unique $t \in T, \underline{h} \in H$ s.t. $t \circ \underline{h} = g$. We suppose that $\lambda_G \in T$.

We now define x in the following way: $\sigma(x_{t\circ\underline{h}}) = y_{\underline{h}}$ and $x_g(h_i) = \phi(g, \underline{h_i})$.

By the definition of x, we have $\Phi(g, x, h_i) = g \circ \underline{h_i}$, therefore we have $\Phi(g, x, h) = g \circ \underline{h}$. In particular $x \in X$.

Now $F(x) = y$. Indeed $F(x)_{\underline{j}} = \sigma(x_{\Phi(\lambda,x,j)}) = \sigma(x_{\lambda\circ\underline{j}}) = y_{\underline{j}}$.

Now let $g \in G$ and write $g^{-1} = t \circ h^{-1}$ for some t, h. Then

$$
\begin{aligned}
F(g \cdot x)_{\underline{j}} &= \sigma((g \cdot x)_{\Phi(\lambda,g\cdot x,j)}) = \sigma(x_{g^{-1}\Phi(\lambda,g\cdot x,j)}) = \sigma(x_{\Phi(g^{-1},x,j)}) = \sigma(x_{g^{-1}\circ\underline{j}}) \\
&= \sigma(x_{t\circ(h^{-1}\underline{j})}) \qquad = y_{h^{-1}\underline{j}} \qquad = (h \cdot y)_{\underline{j}}
\end{aligned}
$$

Therefore $F(g \cdot x) = h \cdot F(x)$.

□

2.3 The proof

We now start from a SFT $X_H \subseteq \Sigma^H$ and build a SFT $X_G \subseteq (\Sigma \times F)^G$ s.t.:

- X_G is empty iff X_H is empty.
- If X_H is weakly aperiodic, then X_G is weakly aperiodic.

Let $\mathcal{P}$ be a finite collections of patterns s.t.

$$X_H = \{y \in \Sigma^H | \forall h \in H, \forall P \in \mathcal{P} \exists j \in \mathrm{Supp}(P), (h \cdot y)_j \neq P_j\}$$

Now we define

$$X_G = \{x \in X | \forall g \in G, \forall P \in \mathcal{P} \exists j \in \mathrm{Supp}(P), F(g \cdot x)_j \neq P_j\}$$

As before, $F(x)_j$ depends only on a finite neighborhood of the identity, thus X_G is a subshift of finite type.

Lemma 2.7. *If X_G is nonempty, then X_H is nonempty. More precisely, for any $x \in X_G$, $F(x) \in X_H$.*

Proof. Obvious by lemma 2.5. □

Lemma 2.8. *If X_H is nonempty, then X_G is nonempty*

Proof. Obvious by lemma 2.6. □

Lemma 2.9. *If X_H is weakly aperiodic, then X_G is weakly aperiodic.*

Proof. Let $x \in X_G$ s.t the orbit of x is finite and $y = F(x)$.

By lemma 2.5, $H \cdot F(x) \subseteq F(G \cdot x)$ therefore the orbit of $F(x)$ is finite. □

We now have proven:

Theorem 3. *Suppose that H is f.p. and acts translation-like on G.*

Then there is an effective procedure that transforms any SFT $X_H \subseteq \Sigma^H$ to a SFT $X_G \subseteq (\Sigma \times F)^G$ s.t.:

- *X_G is empty iff X_H is empty.*
- *If X_H is weakly aperiodic, then X_G is weakly aperiodic.*

In particular:

- *If H has a weakly aperiodic SFT, then so does G.*
- *If H has an undecidable domino problem, then so does G.*

In the following we will need the following refinement of lemma 2.9

Lemma 2.10. *Suppose there exists n s.t. if $x \in X_H$ has finite stabilizer then n divides $[H : Stab(X_H)]$.*

If X_G contains a periodic point, then there exists a subgroup of G of finite index divisible by n.

Proof. Suppose that G has aperiodic point x. That is $Stab(x)$ is of finite index G. Then there exists a normal subgroup $K \subseteq H$ of G which is of finite index. Write $G = AK$. By normality, if $g \in K$ and g' in G then $gg'x = g'x$.

Define $g \sim g'$ iff $\exists h, \phi(g, x, h)g'^{-1} \in K$. It is easy to see that $\sim$ is an equivalence relation on G that factors into an equivalence relation on G/K.

Let $\mathcal{E}$ be an equivalence class on G/K, and g some element of $\mathcal{E}$. Let $H_1 = \{H | \phi(g, x, h) = g\}$.

H_1 is a subgroup of H and by definition $[H : H_1] = |\mathcal{E}|$

Now, let $H_2 = \{h \in H | hF(x) = F(x)\}$. By definition, H_2 contains H_1 (remember that $hF(x) = F(\phi(\lambda, x, h)x)$.

Therefore $|\mathcal{E}| = [H : H_1] = [H : H_2][H_2 : H_1]$.

Thus every equivalence class is of cardinality divisible by n, and therefore G/K is of cardinality divisible by n. □

3 Applications

In this section, we use our main theorem to prove the existence of aperiodic SFTs on some classes of groups.

3.1 Amenability

Theorem 4 ([20])**.** *$\mathbb{F}_2$ acts translation-like on any non amenable f.g. group.*

Corollary 3.1. *Any f.g. non amenable group admits a weakly aperiodic SFT.*

Proof. Piantadosi [17] exhibited a weakly aperiodic SFT on $\mathbb{F}_2$, which is a finitely presented group. ☐

This result can actually be obtained without any reference to translation-like action, or uniformly finite homology [5].

Alternative proof. As G is nonamenable, it admits a (right) paradoxical decomposition, i.e. G can be partitionned into subsets $A_{-n}, A_{-n-1} \dots A_m\}$ s.t.

$$G = \biguplus_{i \geq 0} A_i g_i = \biguplus_{j<0} A_j g_j$$

for some group elements $(g_i)_{-n \leq i \leq m}$ (The use of negative indices while unusual will make the rest of the exposition easier)

Now consider the following subshift

$$\begin{aligned} X &= \left\{ x \in \{-n, \dots m\}^G | \forall g \exists! i \geq 0, x_{gg_i^{-1}} = i \wedge \exists! j > 0, x_{gg_j^{-1}} = j \right\} \\ &= \left\{ x \in \{-n, \dots m\}^G | \forall g \exists! i \geq 0, (g \cdot x)_{g_i^{-1}} = i \wedge \exists! j > 0, (g \cdot x)_{g_j^{-1}} = j \right\} \end{aligned}$$

X is clearly a SFT.

By defining $x_g = j$ if $g \in A_j$ we see that X is nonempty.

Suppose some $x \in X$ has a stabilizer H of finite index in G. Let $N \subseteq H$ be a normal subgroup of G of finite index. Then x induces a paradoxical decomposition of the finite group G/N, a contradiction. ☐

The above example of Piantadosi is linked with a homology group introduced by Chazottes et al.[8]: For a SFT in $\mathbb{Z}^2$ given by Wang tiles[19], one can associate a finite system of linear equations s.t. if X is nonempty, then the system has a nontrivial solution with nonnegative coefficients. This system has as many unknowns as elements of the alphabet, and express the fact that, for $x \in X$, the frequencies of each element of the alphabet in x should satisfy some natural conditions (of course some elements of X do not have frequencies, but frequencies exist μ-almost surely for μ an ergodic measure on X). For example consider the example from Piantadosi, taken in $\mathbb{Z}^2$ with generators a and b, instead of $\mathbb{F}_2$.

$$X = \{x \in \{0,1,2\}^{\mathbb{Z}^2} | x_{ai} = (x_i + 1) \mod 3 \wedge x_i \neq 1 \iff x_{bi} = 1\}$$

Then the system of equations we obtain would be

$$\begin{aligned} z_0 &= z_1 \\ z_1 &= z_2 \\ z_2 &= z_0 \\ z_1 &= z_0 + z_2 \end{aligned}$$

whose only solution is $z_0 = z_1 = z_2 = 0$, ergo X is empty.

Using suitable analogues of the ergodic theorem for amenable groups, it is quite likely that the condition of Chazottes et al. could be generalized to any amenable f.g. group G. However the above example shows this is not true anymore in a nonamenable group.

3.2 Subgroups of finite index

If a point of a subshift in A^G is not aperiodic, it means that his stabilizer is of finite index in G. If every subshift of finite type is periodic, this implies some properties on the lattice of subgroups of finite index. Conversely, we show here that groups for which this lattice behaves badly have aperiodic SFTs.

Proposition 3.2. *A f.g. and not residually finite group admits a weakly aperiodic SFT.*

In particular f.g. simple groups admit weakly aperiodic SFTs.

Proof. As G is not residually finite, there exists a nontrivial element $a \in G$ s.t. every normal subgroup of finite index of G contains a.

$$X = \{x \in \{0,1,2\}^G | \forall g, (g \cdot x)_\lambda \neq (g \cdot x)_a\}$$

X is a SFT and it is easy to see that it is nonempty.

Suppose that X contains a weakly periodic configuration x. By definition of X, $a^{-1} \cdot x \neq x$ therefore the stabilizer of x is a proper subgroup of G of finite index which does not contain a^{-1}, a contradiction ☐

This gives a different proof on the existence of an aperiodic SFT on some Baumslag-Solitar groups.

Using translation-like action, we can say a bit more:

Theorem 5 ([18]). *$\mathbb{Z}$ acts translation-like on any f.g. infinite group.*

Corollary 3.3. *Let G be some infinite f.g. group. Suppose there exists n s.t. G does not contain any subgroup of finite index divisible by n.*

Then G has a weakly aperiodic SFT.

Proof. $\mathbb{Z}$ acts translation-like on G. Now consider

$$X_{\mathbb{Z}} = \{x \in \{0,1,2,\ldots,n-1\}^{\mathbb{Z}} | \forall i, x_{i+1} = x_i + 1 \mod n\}$$

$X_{\mathbb{Z}}$ is a finite subshift where every point is of period exactly n. Now the construction gives a subshift X_G. By lemma 2.10, X_G has no periodic point. ☐

In particular p-groups are group G where for every element $g \in G$, $g^{p^k} = \lambda_G$ for some k. As a consequence, G does not contain any group of finite index divisible by q for any prime $q > p$:

Corollary 3.4. *Infinite f.g. p-groups admit aperiodic SFTs.*

This result is a variation of a construction by Marcinkowski and Nowak [15] of an aperiodic set of tiles on a translation surface on which the Grigorchuk group acts. Using a translation-like action, we are able to obtain directly an aperiodic SFT on the Cayley graph of G.

3.3 Direct products and the Domino Problem

We now give a small example of the relevance of the main theorem to the domino problem.

Lemma 3.5. *If H_1 acts translation-like on G_1 and H_2 acts translation-like on G_2, then $H_1 \times H_2$ acts translation-like on $G_1 \times G_2$*

Corollary 3.6. *For any f.g. infinite G_1, G_2, $G_1 \times G_2$ has a weakly aperiodic SFT and an undecidable domino problem.*

Proof. Under the hypothesis, $\mathbb{Z} \times \mathbb{Z}$ acts on $G_1 \times G_2$. By results of Berger [4], $\mathbb{Z}^2$ has a weakly aperiodic SFT and an undecidable domino problem. ☐

Corollary 3.7. *$\mathbb{Z}^2$ acts translation-like on the Grigorchuk group. Therefore the Grigorchuk group has an aperiodic SFT.*

Proof. G contains a subgroup of finite index of the form $H = H_1 \times H_2$ with H_1, H_2 infinite [16]. As H is of finite index in G, H is finitely generated, therefore H_1 and H_2 are finitely generated as well. ☐

Note that the result of Muchnik and Pak [16] was done in the context of percolation theory: They are interested in the class $\mathcal{S}$ of groups for which there exists $p < 1$ s.t. if every vertex of the (undirected) Cayley graph is activated independently with probability p, then the connected component of the identity is infinite a.s. It can be proven that this property is independent of the generating set so that it is really a property of the group. The Benjamini-Schramm conjecture states that $\mathcal{S}$ is exactly the set of all infinite f.g. groups which are not virtually cyclic, which is the same conclusion as our conjecture. Note that if the Cayley graph of G contains some copy of some Cayley graph of H and $H \in \mathcal{S}$, then $G \in \mathcal{S}$. In particular if H acts translation-like on G and $H \in \mathcal{S}$ then $G \in \mathcal{S}$. We do not know if there is some link between the existence of a weakly aperiodic SFT and percolation.

We will now obtain a better result, that corresponds to our version of the theorem of Ballier and Stein [2]

Lemma 3.8. *Let $\psi : G \mapsto G'$ a onto morphism s.t. $\mathbb{Z}$ acts translation-like on G'. Then there exists a transversal K (i.e. ψ is injective on K and $\psi(K) = G'$) s.t $\mathbb{Z}$ acts translation-like on K.*

(Note that the definition of a translation-like action can be defined even if K is not a group)

Proof. First, $\mathbb{Z}$ acts translation-like on G'. Let $S_{G'}$ be the finite set that witnesses it when the generators of $\mathbb{Z}$ are chosen to be -1 and $+1$. Write $S_{G'} = \{a_1 \dots a_n\}$.

Let $R \subseteq G'$ be a set of representations of the free action, that is every element of G' can be uniquely written $g = r \circ h$ for some $h \in \mathbb{Z}$ and $r \in R$.

Now we fix some notation. A *biinfinite* sequence in a group H is a map $w : \mathbb{Z}^\times \to H$ where $\mathbb{Z}^\times$ is the set of nonzero integers. For such a sequence, we write $w(0) = \lambda$, $w(n) = w_1 \dots w_n$ for $n \geq 0$ and $w(n) = w_{-1}w_{-2} \dots w_n$ for $n < 0$,

Now let $r \in R$. As $r \circ n = r \circ 1 \circ 1 \circ \dots \circ 1$ for $n \geq 0$ (and similarly for $n < 0$), there exists an infinite sequence w^r with values in $S_{G'}$ s.t. $r \circ n = rw^r(n)$. Basically, an element of $r \circ \mathbb{Z}$ is represented by the label on the path from r to this element following the copy of the Cayley graph of $\mathbb{Z}$. Note that there is a unique path from r to this element, as the Cayley graph of $\mathbb{Z}$ is a tree.

Now we take representatives of all elements of $S_{G'}$ and all elements of R: Let $\theta : R \cup S_{G'} \to G$ s.t. $\psi\theta(g) = g$ whenever it is defined.

Now we take $K = \{\theta(r)\theta(w^r)(n) | r \in R, n \in \mathbb{Z}\}$, where $\theta(w^r)_i = \theta(w^r_i)$

It is easy to see that K is a transversal and that all elements in the above formula are distinct.

Furthermore $\mathbb{Z}$ acts on K by $\theta(r)\theta(w^r(n)) \circ m = \theta(r)\theta(w^r(n+m))$. This action is clearly free.

Furthermore, for all $k \in K$, $k \circ 1 = kb$ for some $b \in \theta(S_{G'}) \cup \theta(S_{G'})^{-1}$, and the same is true for -1, so that $\theta(S_{G'}) \cup \theta(S_{G'})^{-1}$ witnesses the fact that $\mathbb{Z}$ acts translation-like on K. □

The same could be done replacing $\mathbb{Z}$ by any group which admits a Cayley graph which is a tree, i.e. a free group. The result is false in general if $\mathbb{Z}$ is replaced by a group which is not free, take for example the morphism from the free group to $\mathbb{Z}^2$.

Corollary 3.9. *If $H \subseteq Z(G)$ is finitely generated and G/H is f.g. infinite, then $\mathbb{Z} \times H$ acts translation-like on G.*

Proof. Use the previous lemma and define $ka \circ (n, b) = (k \circ n)(ab)$ for $k \in K, a, b \in H$ and $n \in \mathbb{Z}$. It is easy to see that it is indeed an action, and it is translation-like □

Corollary 3.10 ([2])**.** *If $Z(G)$ is f.g., $\mathbb{Z} \subseteq Z(G)$ and $G/Z(G)$ is f.g. infinite, then G has a weakly aperiodic SFT and an undecidable domino problem*

Corollary 3.11. *If a f.g. group G admits $\mathbb{Z}$ as a normal subgroup, then it has undecidable domino problem unless it is virtually cyclic.*

More precisely, some Baumslag Solitar group $B(1,n)$, $n \neq 0$, acts translation-like on G.

Proof. Let $H \subseteq G$ with $H \simeq \mathbb{Z}$ and let a be the generator of H. As H is normal, for any element t of G, we have $tat^{-1} = a^n$ for some n that depends on t.

If $tat^{-1} = a^n$ for some $n \notin \{1, -1\}$, then the subgroup generated by t and a is isomorphic to the Baumslag Solitar group $B(1,n)$ and therefore G admits a subgroup with undecidable domino problem and therefore has undecidable domino problem.

Otherwise $tat^{-1} = a^{\pm 1}$ for all $t \in G$. Then $G' = \{t | tat^{-1} = a\}$ is a subgroup of G that contains H of index at most 2 in G. H is in the center of G', so that $\mathbb{Z} \times H \simeq \mathbb{Z}^2$ acts translation-like on G', therefore it acts translation like on G. □

4 Further generalizations

First, let remark that the whole construction works as well starting from a finitely presented monoid rather than a finitely presented group. In fact, the result of Ballier and Stein [2] build a $\mathbb{Z} \times \mathbb{N}$-action on any f.g nilpotent group which is not virtually $\mathbb{Z}$. We do not go into details on this generalization as some details are quite cumbersome. In particular a translation-like action by $\mathbb{N}$ does not partition the Cayley graph into copies of $\mathbb{N}$ but into some kind of trees.

Note that all our proofs exploit the aperiodicity of X_H to prove that X_G is aperiodic. There is another way to do this, by forcing the copies of H inside G to be infinite.

For example, for a finite subset $\mathcal{T}$ of $S^\star \times S^\star$, let's consider the following variant of X

$$X = \left\{ x \in F^G | \forall g \in G, \begin{array}{c} \forall (h,h') \in \mathcal{R}, \Phi(g,x,h) = \Phi(g,x,h') \\ \forall (h,h') \in \mathcal{T}, \Phi(g,x,h) \neq \Phi(g,x,h') \end{array} \right\}$$

Isolated groups [10] (also called groups with finite absolute presentation) admit a presentation in this term: If G is an isolated group with generators S, there exists a finite set of relations $\mathcal{R}$ and antirelations $\mathcal{T}$ s.t. G is the only f.g. group generated by S that satisfy these relations and antirelations. As a consequence, any copy of H (that is $\{\Phi(g,x,h), h \in H\}$ for some g) should be in bijection with H, and therefore infinite.

We do not give more details as all these (infinite) groups are not residually finite, therefore this result is subsumed by the construction of an aperiodic SFT on any nonresidually finite group.

An other interesting direction is the undecidability of the *periodic domino problem*: decide, given a SFT X over a group G, if X is weakly aperiodic. Most of our work here fail for this problem: it is not true that X_G is weakly aperiodic iff X_H is.

5 Open Problems

Carroll and Penland [6] showed that the existence of an aperiodic SFT is also closed under another notion of containment: If H is a finitely generated normal subgroup of G and G/H has a weakly aperiodic SFT (resp. has an undecidable domino problem), then G does. Finite generation of H is essential here: $\mathbb{F}_2$ contains $\mathbb{Z}^2$ as a quotient, but $\mathbb{Z}^2$ has an undecidable word problem and $\mathbb{F}_2$ does not.

Can this be generalized as well by asking for less than a quotient ?

To tackle the two conjectures stated above, we introduce now a new conjecture:

Conjecture 3. *If G is a nontrivial f.g. group, then some nontrivial one-relator group acts translation-like on G.*

Here nontrivial means not virtually cyclic. Note that the amenable nontrivial one-relator groups are the Baumslag Solitar groups, so that this conjecture may be restated as: If G is a nontrivial amenable group, then some Baumslag-Solitar group acts translation-like on G.

This conjecture would imply that every nontrivial group admits a weakly aperiodic SFT, and the Benjamini-Schramm conjecture stated above.

We give the lamplighter group as a potential counterexample to the three conjectures of this paper.

References

[1] Nathalie Aubrun and Jarkko Kari. Tiling Problems on Baumslag-Solitar groups. In *Machines, Computations and Universality (MCU)*, number 128 in Electronic Proceedings in Theoretical Computer Science, pages 35–46, 2013.

[2] Alexis Ballier and Maya Stein. The domino problem on groups of polynomial growth. arXiv:1311.4222.

[3] Robert Berger. *The Undecidability of the Domino Problem.* PhD thesis, Harvard University, 1964.

[4] Robert Berger. *The Undecidability of the Domino Problem.* Number 66 in Memoirs of the American Mathematical Society. The American Mathematical Society, 1966.

[5] Jonathan Block and Shmuel Weinberger. Aperiodic Tilings, Positive Scalar Curvature, and Amenability of Spaces. *Journal of the American Mathematical Society*, 5(4):907–918, 1992.

[6] David Carroll and Andrew Penland. Periodic Points on Shifts of Finite Type and Commensurability Invariants of Groups. arXiv:1502.03195.

[7] Tullio Ceccherini-Silberstein and Michel Coornaert. *Cellular Automata on Groups*. Springer Monographs in Mathematics. Springer, 2010.

[8] Jean-René Chazottes, Jean-Marc Gambaudo, and François Gautero. Tilings of the plane and Thurston semi-norm. *Geometriae Dedicata*, 173(1):129–142, 2014.

[9] David Bruce Cohen. The large scale geometry of strongly aperiodic subshifts of finite type. arXiv:1412.4572, 2014.

[10] Yves de Cornulier, Luc Guyot, and Wolfgang Pitsch. On the isolated points in the space of groups. *Journal of Algebra*, 307:254–277, 2007.

[11] Tullia Dymarz. Bilipschitz equivalence is not equivalent to quasi-isometric equivalence for finitely generated groups. *Duke Mathematical Journal*, 154(3):509–526, 2010.

[12] Jarkko Kari. A small aperiodic set of Wang tiles. *Discrete Mathematics*, 160:259–264, 1996.

[13] Jarkko Kari. On the undecidability of the tiling problem. In *Current Trends in Theory and Practice of Computer Science (SOFSEM)*, pages 74–82, 2008.

[14] Dietrich Kuske and Markus Lohrey. Logical aspects of Cayley-graphs: the group case. *Annals of Pure and Applied Logic*, 131(1-3):263–286, January 2005.

[15] Michał Marcinkowski and Piotr W. Nowak. Aperiodic tilings of manifolds of intermediate growth. *Groups, Geometry, and Dynamics*, 8(2):479–483, 2014.

[16] Roman Muchnik and Igor Pak. Percolation on Grigorchuk Groups. *Communications in Algebra*, 29(2):661–671, 2001.

[17] Steven T. Piantadosi. Symbolic Dynamics on Free Groups. *Discrete and Continuous Dynamical Systems*, 20(3):725–738, March 2008.

[18] Brandon Seward. Burnside's Problem, Spanning Trees and Tilings. *Geometry and Topology*, 18:179–210, 2014.

[19] Hao Wang. Proving theorems by Pattern Recognition II. *Bell Systems technical journal*, 40:1–41, 1961.

[20] Kevin Whyte. Amenability, Bilipschitz equivalence, and the Von Neumann Conjecture. *Duke Mathematical Journal*, 99(1):93–112, 1999.